

MEASURABILITY AND PERFECT SET THEOREMS FOR EQUIVALENCE RELATIONS WITH SMALL CLASSES

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ABSTRACT. We ask whether Δ_2^1 or Σ_2^1 equivalence relations with I -small classes for I a σ -ideal must have perfectly many classes. We show that for a wide class of ccc σ -ideals, a positive answer for Δ_2^1 equivalence relations is equivalent to the I -measurability of Δ_2^1 sets. However, the analogous statement for Σ_2^1 equivalence relations is false: Σ_2^1 equivalence relations with meager classes have a perfect set of pairwise inequivalent elements if and only if Δ_2^1 sets have the Baire property.

1. INTRODUCTION

An equivalence relation E on a Polish space X is said to have *perfectly many classes* if there is a perfect set $P \subseteq X$ whose elements are pairwise inequivalent.

Given a σ -ideal I , we say that a set A is *I -small* if $A \in I$ and *I -positive* if $A \notin I$. \mathbb{P}_I is the partial order of I -positive Borel sets, ordered by inclusion. I is *proper* if \mathbb{P}_I is.

A theorem due to Silver states that Π_1^1 equivalence relations either have countably many classes or perfectly many classes. Therefore, a Π_1^1 equivalence relation on a Borel I -positive set whose classes are I -small must have perfectly many classes – a property we will denote by $PSP_I(\Pi_1^1)$:

Definition 1.1. For I a σ -ideal and Γ a pointclass, $PSP_I(\Gamma)$ (for “Perfect Set Property”) is the following statement: “If $E \in \Gamma$ is an equivalence relation on a Borel I -positive set with I -small classes then E has perfectly many classes”.

Silver’s theorem proving $PSP_I(\Pi_1^1)$ relied on the definition of the equivalence relation – namely, its Π_1^1 definition. Other theorems use the measurability of the equivalence relation to arrive at the same conclusion:

Theorem 1.2. [19] *If E is an equivalence relation on a Borel nonmeager set that has the Baire property, and all E -classes are meager, then E has perfectly many classes.*

Theorem 1.3. [20] *If E is a Lebesgue measurable equivalence relation on a Borel set of positive measure and all E -classes are null, then E has perfectly many classes.*

In particular, since analytic sets are Lebesgue measurable and have the Baire property, Mycielski has shown $PSP_{meager}(\Sigma_1^1)$ and $PSP_{null}(\Sigma_1^1)$. Furthermore, in [8] we have shown that $PSP_I(\Sigma_1^1)$ is true for any proper σ -ideal I .

This paper investigates $PSP_I(\Delta_2^1)$ and $PSP_I(\Sigma_2^1)$. We can use Mycielski’s results above as a starting point – they clearly imply:

Observation 1.4. (1) *If all Δ_2^1 sets are Lebesgue measurable (have the Baire property) then $PSP_{null}(\Delta_2^1)$ ($PSP_{meager}(\Delta_2^1)$).*
 (2) *If all Σ_2^1 sets are Lebesgue measurable (have the Baire property) then $PSP_{null}(\Sigma_2^1)$ ($PSP_{meager}(\Sigma_2^1)$).*

In section 2 we will see how measurability can be generalized to any σ -ideal. Then in light of the above observation we ask:

Problem 1.5. Let I be a σ -ideal.

- (1) Is I -measurability of all Δ_2^1 sets equivalent to $PSP_I(\Delta_2^1)$?
- (2) Is I -measurability of all Σ_2^1 sets equivalent to $PSP_I(\Sigma_2^1)$?

1.1. Previous results. The main results on $PSP_I(\Delta_2^1)$ and $PSP_I(\Sigma_2^1)$ in [8] are:

Theorem 1.6. *Let I be a proper σ -ideal, and assume Σ_3^1 \mathbb{P}_I -generic absoluteness. Then $PSP_I(\Delta_2^1)$.*

Theorem 1.7. *(countable ideal) The following are equivalent:*

- (1) $PSP_{countable}(\Delta_2^1)$.
- (2) For any real z , $\mathbb{R}^{L[z]} \neq \mathbb{R}$.

Theorem 1.8. *(meager ideal) If for any real z there is a Cohen real over $L[z]$, then $PSP_I(\Sigma_2^1)$.*

Theorem 1.7 serves as a positive evidence for problem 1.5 (1): in [5] 7.1 it is shown that "for any real, $\mathbb{R}^{L[z]} \neq \mathbb{R}$ " is equivalent to "all Δ_2^1 sets are measurable with respect to the countable ideal".

We note that in [8] the notion of PSP_I refers to equivalence relations on all reals, whereas here it is a stronger notion referring to equivalence relations on Borel I -positive sets. However, all statements and proofs of [8] are valid for the stronger notion considered here, with the obvious changes in the proofs.

1.2. Measurability, Generic Absoluteness and Transcendence over L . Judah and Shelah [12] have shown that Δ_2^1 sets are Lebesgue measurable if and only if for every z there is a random real over $L[z]$. In [3] it is shown that Lebesgue measurability of Δ_2^1 sets is equivalent to Σ_3^1 random real generic absoluteness, which is, Σ_3^1 statements are preserved under random real forcing.

The above results indicate a connection between measurability of Δ_2^1 sets, Σ_3^1 -generic-absoluteness and transcendence over L - namely, existence of generics over L . This connection is not reserved to the case of random real forcing - it exists for Cohen forcing where measurability is replaced by the Baire property and random reals by Cohen reals. It also exists for Sacks forcing [13] with the appropriate generalizations of the notions of measurability and genericity. In fact, Brendle and Lowe [5] find similar equivalences for most of the better known examples, whereas Ikegami in [14] shows how to extend the above results to a wide class of proper σ - ideals.

The notion of I -measurability for a general σ -ideal is discussed in section 2. Here we list Ikegami's results, after translating them to the context of I -measurability we are working with in this paper.

A σ -ideal I is said to be Σ_n^1 or Π_n^1 if the set of Borel codes of I -small sets is. The term "provably ccc" refers to σ -ideals which are ccc in all models of ZFC . An ideal is said to be *Borel generated* if I -small sets are contained in I -small Borel sets. Σ_3^1 - \mathbb{P}_I -generic-absoluteness is the property that Σ_3^1 statements on ground model reals are absolute between the universe and \mathbb{P}_I -generic extensions of the universe.

For a forcing notion \mathbb{P} , we say that \mathbb{P} is *strongly arboreal* if the conditions of \mathbb{P} are perfect trees on ω , and

$$T \in \mathbb{P}, s \in T \Rightarrow T \restriction_s \in \mathbb{P}$$

where $T \restriction_s = \{t : t \in T; t \supseteq s \text{ or } t \subseteq s\}$.

Theorem 1.9. [14] *Let I be a provably ccc, provably Δ_2^1 and Borel generated σ -ideal such that \mathbb{P}_I is strongly arboreal. The following are equivalent:*

- (1) Every Δ_2^1 set of reals is I -measurable.
- (2) For any real z and $B \in \mathbb{P}_I$, there is an $L[z]$ generic in B .
- (3) Σ_3^1 - \mathbb{P}_I -generic-absoluteness.

Theorem 1.10. [14] *Let I be a provably ccc, provably Δ_2^1 and Borel generated σ -ideal such that \mathbb{P}_I is strongly arboreal. The following are equivalent:*

- (1) Every Σ_2^1 set of reals is \mathbb{P}_I -measurable.
- (2) For any real z , the set of \mathbb{P}_I generics over $L[z]$ is co- I .

1.3. The results of this paper. We devote section 2 for a detailed exposition of the notion of I -measurability, where I is any σ -ideal:

Definition 1.11. [5] Let I be a σ -ideal, and $A \subseteq \mathbb{R}$. We say that A is I -measurable if for every $B \in \mathbb{P}_I$, there is $B' \subseteq B$ Borel I -positive such that either $B' \subseteq A$ or $B' \subseteq^{\sim} A$.

This notion extends the notion of measurability for ccc σ -ideals, as the following proposition shows. Recall that I is *Borel generated* if any I -small set is contained in a Borel I -small set.

Proposition 1.12. *If I is ccc and Borel generated, then A is I -measurable if and only if there is B Borel such that $A \triangle B \in I$.*

A few basic facts on regularity of measurable sets are given, among which:

Proposition 1.13. *If A is universally Baire and I is a proper σ -ideal, then A is I -measurable.*

Proposition 1.14. *If there is a measurable cardinal and I is a proper σ -ideal, then Σ_2^1 sets are I -measurable.*

We say that I is *provably ccc* if “ I is ccc” is a theorem of ZFC . We say that I is Σ_n^1 or Π_n^1 if

$$\{c : c \in BC, B_c \in I\}$$

is Σ_n^1 or Π_n^1 , where BC is the set of Borel codes and for $c \in BC$, B_c is the Borel set coded by c .

Proposition 1.15. *Let I be a Σ_2^1 provably ccc σ -ideal. If ω_1 is inaccessible to the reals, then Σ_2^1 sets are I -measurable.*

In section 3 we elaborate on equivalent formulations of measurability in terms of generic absoluteness and transcendence properties over L . This section is heavily based on ideas, proofs and arguments from Ikegami [13] and [14], presented in a somewhat different context. Establishing those equivalences in the context we work in will prove useful in understanding the perfect set properties of equivalence relations with small classes. An overview of Ikegami’s original results can be found in subsection 1.2 above.

Recall that Σ_3^1 - \mathbb{P}_I -generic-absoluteness is the property that Σ_3^1 statements on ground model reals are absolute between the universe and \mathbb{P}_I -generic extensions of the universe.

We use the following notion due to Zapletal to establish an equivalence between Σ_3^1 - \mathbb{P}_I -generic absoluteness and Δ_2^1 I -measurability for any proper σ -ideal I :

Definition 1.16. ([23] 2.3.4) For Γ a pointclass, we say that Γ has \mathbb{P}_I -Borel uniformization if given $A \in \Gamma$ a subset of $(\omega^\omega)^2$ with nonempty sections and $B \subseteq \omega^\omega$ I -positive, there is $B' \subseteq B$ Borel I -positive and a Borel function $f \subseteq A$ with domain B' .

Theorem 1.17. *Let I be a proper σ -ideal. The following are equivalent:*

- (1) $\Sigma_3^1\text{-}\mathbb{P}_I\text{-generic-absoluteness}$.
- (2) Σ_2^1 has \mathbb{P}_I -Borel uniformization.
- (3) Π_1^1 has \mathbb{P}_I -Borel uniformization.
- (4) Δ_2^1 sets are I -measurable.

For definable enough provably ccc σ -ideals, an argument from [14] adds transcendence over L to the list of equivalent statements:

Theorem 1.18. *Let I be a Σ_2^1 provably ccc σ -ideal. Then the following are equivalent:*

- (1) Δ_2^1 sets are I -measurable.
- (2) For every $B \in \mathbb{P}_I$ and for every real z , there is a \mathbb{P}_I -generic over $L[z]$ in B .

Sections 4 and 5 focus on $PSP_I(\Delta_2^1)$, $PSP_I(\Sigma_2^1)$ and problem 1.5. Section 4 presents properties of transcendence over L which are sufficient conditions for $PSP_I(\Sigma_2^1)$ when I is provably ccc and definable enough. Section 5 completes the picture by providing necessary conditions for both $PSP_I(\Sigma_2^1)$ and $PSP_I(\Delta_2^1)$ for the same class of σ -ideals.

A set A of reals is a set of $\mathbb{P}_I * \dot{\mathbb{P}}_I$ generics if for every $x \in A$ and $y \in A$ which are not equal, (x, y) is $\mathbb{P}_I * \dot{\mathbb{P}}_I$ generic.

Theorem 1.19. *Let I be Σ_2^1 or Π_2^1 and provably ccc. If for any real z and $B \in \mathbb{P}_I$ there is a perfect set $P \subseteq B$ of $\mathbb{P}_I * \dot{\mathbb{P}}_I$ generics over $L[z]$, then $PSP_I(\Sigma_2^1)$.*

Then together with another result of [4] on the existence of a perfect set of $\mathbb{P}_I * \dot{\mathbb{P}}_I$ generics, we have:

Corollary 1.20. *Let I be Σ_2^1 or Π_2^1 , provably ccc, homogeneous and with the Fubini property. If ω_1 is inaccessible to the reals then $PSP_I(\Sigma_2^1)$.*

We remark that stronger large cardinal assumptions clearly imply $PSP_I(\Sigma_2^1)$ and more. For example, if a measurable cardinal exists, then for any proper σ -ideal I , $PSP_I(\Sigma_3^1)$ and $PSP_I(\Pi_3^1)$ are true. To see why, reread section 2 of [8] and replace the last line of the proof of claim 2.6 by the argument from [7] theorem 3.9.

As to the necessary conditions:

Theorem 1.21. *Let I be Σ_2^1 and provably ccc. $PSP_I(\Sigma_2^1)$ implies that for every $B \in \mathbb{P}_I$ and for every real z there exists a perfect set $P \subseteq B$ of \mathbb{P}_I -generics over $L[z]$.*

Theorem 1.22. *Let I be Σ_2^1 and provably ccc. $PSP_I(\Delta_2^1)$ implies that for every $B \in \mathbb{P}_I$ and for every real z there exists a \mathbb{P}_I -generic over $L[z]$ in B .*

At that point we can answer problem 1.5 (1) positively for a wide class of ccc σ -ideals:

Corollary 1.23. *Let I be Σ_2^1 and provably ccc. The following are equivalent:*

- (1) $PSP_I(\Delta_2^1)$.
- (2) Δ_2^1 sets are I -measurable.

However, problem 1.5 (2) has a negative answer:

Corollary 1.24. *The following are equivalent:*

- (1) $PSP_{meager}(\Sigma_2^1)$.

(2) Δ_2^1 sets have the Baire property.

Trying to characterize $PSP_{null}(\Sigma_2^1)$ leads to an open problem of Brendle ([4] 2.8): is the existence of a perfect set of random reals equivalent to the existence of a perfect set of mutually random reals? A positive answer will imply that $PSP_{null}(\Sigma_2^1)$ is equivalent to the existence of a perfect set of random reals over $L[z]$ for any z .

Another problem yet to be solved is characterizing $PSP_{countable}(\Sigma_2^1)$. In light of theorem 1.7 and [5] 7.1, we conjecture:

Problem 1.25. If for every z there is $x \notin L[z]$, then $PSP_{countable}(\Sigma_2^1)$.

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2. I -MEASURABLE SETS

Definition 2.1. Let I be a σ -ideal, and $A \subseteq \mathbb{R}$. We say that A is I -measurable if for every $B \in \mathbb{P}_I$, there is $B' \subseteq B$ Borel I -positive such that either $B' \subseteq A$ or $B' \subseteq \sim A$.

Remark 2.2. The following are easy to observe:

- (1) Borel sets are measurable.
- (2) There is a non measurable set.
- (3) If A is I -measurable, then either A or $\sim A$ contain a Borel I -positive set.
- (4) If I is such that I -positive sets contain I -positive Borel sets, then all I -small sets are measurable. In that case, a set A will be I small if and only if for every $B \in \mathbb{P}_I$ there is $B' \subseteq B$ in \mathbb{P}_I such that $B' \subseteq \sim A$.

Proposition 2.3. If I is ccc and Borel generated, then A is I -measurable if and only if there is B Borel such that $A \triangle B \in I$. Therefore, for I ccc and Borel generated, the I -measurable sets form a σ -algebra.

Proof. First assume there is a Borel set B such that $A \triangle B \in I$, and fix $C \in I$ such that $A \triangle B \subseteq C$. Given a condition D , $D - C$ is I positive and disjoint of $A \triangle B$. Hence, $(D - C) \cap B \subseteq A$ and $(D - C) \cap (\sim B) \subseteq \sim A$.

For the other direction, let A be I -measurable. The set

$$D = \{B : B \in \mathbb{P}_I; B \subseteq A \text{ or } B \subseteq \sim A\}$$

is dense – let B_n be a maximal antichain of elements of D . Define

$$B = \bigcup_{B'_n \subseteq A} B'_n$$

and

$$C = \bigcup_{B'_n \subseteq \sim A} B'_n.$$

The complement $\sim(B \cup C)$ must be I -small, otherwise we could extend the maximal antichain. We then claim that B is the required approximation of A , since $B \subseteq A$ and

$$A - B \subseteq \sim(B \cup C)$$

which is I -small. □

The last proposition shows that definition 2.1 coincides with the traditional definition for a wide class of ccc ideals. For many other examples of σ -ideals, definition 2.1 is not new as well. For example, for the case of the countable ideal the notion of Sacks measurability is well known for years. A set $A \subseteq \mathbb{R}$ is Sacks measurable if for any perfect set P there is $P' \subseteq P$ perfect such that $P' \subseteq A$ or $P' \subseteq^{\sim} A$ – exactly the same as the definition discussed here.

We list related regularity properties of universally Baire and projective sets. $M \preceq H_\theta$ always mean that M is a countable elementary submodel of a large enough H_θ .

Proposition 2.4. *If A is universally Baire and I is a proper σ -ideal, then A is I -measurable. Furthermore, some $B \in \mathbb{P}_I$ forces $x_{gen} \in A$ if and only if A contains a Borel I -positive subset.*

Proof. Let $B \in \mathbb{P}_I$ and let $A, B \in M \preceq H_\theta$ a countable elementary submodel of a large enough H_θ . Let $B' \subseteq B$ be such that $B' \Vdash x_{gen} \in A$ or $B' \Vdash x_{gen} \notin A$. Let $B'' \subseteq B'$ be the set of M -generics. Using the universally Baire definition of A we find that $B'' \subseteq A$ or $B'' \subseteq^{\sim} A$. □

Corollary 2.5. *Let A be universally Baire and I a ccc σ -ideal. Then A is either contained in a Borel I -small set, or contains a Borel I -positive set.*

Proof. If some condition forces $x_{gen} \in A$, then by the last proposition, A contains a Borel I -positive set. Otherwise, $\Vdash_{\mathbb{P}_I} x_{gen} \in^{\sim} A$. Hence the M -generics are all in $^{\sim}A$, so $^{\sim}A$ contains a Borel co- I set, and A is contained in a Borel I -small set. □

Proposition 2.6. *If there is a measurable cardinal and I is a proper σ -ideal, then Σ_2^1 sets are I -measurable.*

Proof. Let A be Σ_2^1 and $B \in \mathbb{P}_I$. We may assume that $B \Vdash x_{gen} \in A$ or $B \Vdash x_{gen} \notin A$. Now let $M \preceq H_\theta$ contain all the relevant information and the measurable cardinal. Let $B' \subseteq B$ be the set of M -generics in B . There are 2 cases:

- $B \Vdash x_{gen} \in A$: We show that $B' \subseteq A$. Indeed, let $x \in B'$. Then $M[x] \models x \in A$, hence $x \in A$.
- $B \Vdash x_{gen} \notin A$: We show that $B' \subseteq^{\sim} A$. Indeed, let $x \in B'$. Then

$$M[x] \models x \notin A.$$

We claim that $x \notin A$. The argument is as in [7] theorem 3.9. Assume otherwise – $x \in A$ – so there is some $\alpha < \omega_1$ such that inner models in which α is countable think that $x \in A$. For ease of notation, we let $N = M[x]$, and iterate N uncountably many times, so that N_{ω_1} will contain all countable ordinals. In $N_{\omega_1}[\text{coll}(\omega, \alpha)]$, $x \in A$, and using Shoenfield's absoluteness,

$$N_{\omega_1} \models x \in A$$

as well. But N_{ω_1} is an elementary extension of $N = M[x]$, in contradiction with $M[x] \models x \notin A$. □

Proposition 2.7. *Let I be a Σ_2^1 provably ccc σ -ideal. If ω_1 is inaccessible to the reals, then Σ_2^1 sets are I -measurable.*

Proof. Let A be Σ_2^1 , and $B \in \mathbb{P}_I$. Extend B to B' forcing $x_{gen} \in A$ or $x_{gen} \notin A$. The first case is exactly the same as the first case in proposition 2.6. For the second case, let $M \preceq H_\theta$ be a countable elementary

submodel containing all the relevant information and $L_{\omega_1^L}$, and in particular containing all constructible reals. It will be enough to show that the M -generics in B' are elements of $\sim A$. Indeed, if

$$M[x] \models x_{gen} \in \sim A$$

then

$$L_{\omega_1^L}[x] \models x_{gen} \in \sim A$$

by analytic absoluteness only. Since x is generic over L (using the assumptions on I), $\omega_1^L = \omega_1^{L[x]}$ and we can use Shoenfield's absoluteness to reflect the last statement to \mathbb{V} and complete the proof. \square

Remark 2.8. In fact, a sufficient assumption on the σ -ideal I is that for every z , $I \cap L[z] \in L[z]$, and $L[z] \models I \cap L[z]$ is ccc.

Proposition 2.9. *Let A be Σ_2^1 and I a proper σ -ideal. If some $B \in \mathbb{P}_I$ forces $x_{gen} \in A$ then A contains a Borel I -positive subset.*

Proof. A can be represented as a union of \aleph_1 Borel sets:

$$A = \bigcup_{\alpha < \omega_1} B_\alpha.$$

Let $C \Vdash x_{gen} \in A$. Then there is $C' \subseteq C$ such that $C' \Vdash x_{gen} \in B_\alpha$ – where we have used the assumption that ω_1 is preserved. B_α then must be I -positive. \square

The rest of this section is concerned only with ccc σ -ideals. Both of the following are false for general proper σ -ideals – consider the countable ideal and the Π_1^1 set with no perfect subset.

Proposition 2.10. *Let A be Π_2^1 and I a ccc σ -ideal. If A is I -positive then there is some $B \in \mathbb{P}_I$ forcing $x_{gen} \in A$.*

Proof. Assume

$$\mathbb{P}_I \Vdash x_{gen} \notin A.$$

As before, $\sim A = \bigcup_{\alpha < \omega_1} B_\alpha$. Find a maximal antichain forcing $x_{gen} \in B_0$, extend it to a maximal antichain forcing $x_{gen} \in B_0 \cup B_1$, and so on. Since antichains are countable, the process must stop at a countable level. The union of all conditions in that antichain is a Borel set B contained modulo I in $\sim A$ such that

$$\mathbb{P}_I \Vdash x_{gen} \in B.$$

The complement of B must then be I -small. A is contained in $\sim B$ modulo I , therefore it is I -small as well – which is what we wanted to show. \square

Corollary 2.11. *Let A be Δ_2^1 and I a ccc σ -ideal. If A is I -positive then A contains a Borel I -positive subset. In particular, a coanalytic set with no perfect subset is I -small with respect to any ccc σ -ideal.*

Proof. Follows of the last two propositions. \square

3. MEASURABILITY, GENERIC ABSOLUTENESS AND TRANSCENDENCE OVER L

The main result of the following section establishes equivalences between three notions: generic absoluteness, measurability – as discussed in the previous section – and the following notion due to Zapletal.

Definition 3.1. ([23] 2.3.4) For Γ a pointclass, we say that Γ has \mathbb{P}_I -Borel uniformization if given $A \in \Gamma$ a subset of $(\omega^\omega)^2$ with nonempty sections and $B \subseteq \omega^\omega$ I -positive, there is $B' \subseteq B$ Borel I -positive and a Borel function $f \subseteq A$ with domain B' .

Theorem 3.2. Let I be a proper σ -ideal. The following are equivalent:

- (1) Σ_3^1 - \mathbb{P}_I -generic-absoluteness.
- (2) Σ_2^1 has \mathbb{P}_I -Borel uniformization.
- (3) Π_1^1 has \mathbb{P}_I -Borel uniformization.
- (4) Δ_2^1 sets are I -measurable.

Proof. (1) \Rightarrow (2): Let A be Σ_2^1 with nonempty sections – $\forall x \exists y (x, y) \in A$ – which is Σ_3^1 and hence by assumption is preserved in \mathbb{P}_I generic extensions. Then there is a name τ such that

$$B \Vdash (x, \tau) \in A.$$

Let $f : C \rightarrow Y$ be a Borel function in the ground model such that $C \Vdash (x, f(x)) \in A$. If $M \preceq H_\theta$ is a countable elementary submodel containing all the relevant information and x is M generic, then $M[x] \models (x, f(x)) \in A$ and using Π_1^1 absoluteness, $(x, f(x)) \in A$.

(3) \Rightarrow (4) : Let $B \in \mathbb{P}_I$ and A a Δ_2^1 set. Fix C and D Π_1^1 subsets of the plane such that $\Pi(C)$, the projection of C , is A , and $\Pi(D) =^\sim A$. Since $C \cup D$ is a Π_1^1 set with nonempty sections, there is $B' \subseteq B$ in \mathbb{P}_I and f a Borel function such that

$$\forall x \in B' : (x, f(x)) \in C \cup D.$$

It follows that for $x \in B'$:

$$x \in A \Leftrightarrow (x, f(x)) \in C \Leftrightarrow (x, f(x)) \notin D,$$

so $B' \cap A$ is Borel. The same argument works for $B' \cap^\sim A$. One of $B' \cap A, B' \cap^\sim A$ must be I -positive.

(4) \Rightarrow (1): We use the notation of [14] and follow the proof of [14] theorem 4.1 and claim 4.2.

Assume all Δ_2^1 sets are I -measurable. We show that all Δ_2^1 sets are \mathbb{P}_I -Baire, and that will be enough (See [14] 3.9 and [9]).

Let $f : st(\mathbb{P}_I) \rightarrow \omega^\omega$ be a Baire measurable function and A a Δ_2^1 set. It will be enough to show that

$$\{B : O_B \cap f^{-1}(A) \text{ meager or } O_B - f^{-1}(A) \text{ meager}\}$$

is a dense set in \mathbb{P}_I , where O_B is $\{G \in st(\mathbb{P}_I) : B \in G\}$. Indeed, let $B \in \mathbb{P}_I$. There is a name τ such that for comeagerly many $G \in st(\mathbb{P}_I)$:

$$f(G) = \tau[G].$$

Since I is proper, there is $B' \subseteq B$ in \mathbb{P}_I and $g : B' \rightarrow \omega^\omega$ Borel such that

$$B' \Vdash g(x_G) = \tau[G],$$

which means that for comeagerly many $G \in st(\mathbb{P}_I)$ such that $B' \in G$,

$$g(x_G) = \tau[G] = f(G).$$

Since $g^{-1}(A)$ is Δ_2^1 , it is measurable by our assumption. Let $B'' \subseteq B'$ in \mathbb{P}_I be such that

$$B'' \subseteq g^{-1}(A)$$

or

$$B'' \subseteq g^{-1}(\sim A).$$

We continue with the 1st case – the 2nd is similar. Since $B'' \in G$ implies $B' \in G$, we conclude that for comeagerly many $G \in st(\mathbb{P}_I)$ such that $B'' \in G$

$$g(x_G) = \tau[G] = f(G) \in A$$

whereas $f(G) \in A$ because $x_G \in B''$ and $B'' \subseteq g^{-1}(A)$. That shows that $O_{B''} - f^{-1}(A)$ is meager. \square

We give here another argument for $(3) \Rightarrow (1)$ which we find interesting on its own. It is based on an argument from the proof of [13] theorem 3.1:

$(3) \Rightarrow (1)$: By way of contradiction, assume $\forall x \neg \Psi(x)$ but $\Vdash \exists x \Psi(x)$, where $\Psi(x) = \forall y \Phi(x, y)$ and Φ is Σ_1^1 . Fix $B \in \mathbb{P}_I$ and $f \in \mathbb{V}$ a Borel function such that

$$B \Vdash \Psi(f(x_{gen})).$$

In \mathbb{V}

$$\forall x \exists y \neg \Phi(f(x), y)$$

so we can use Π_1^1 \mathbb{P}_I -Borel uniformization to produce a Borel function $g : B' \rightarrow \omega^\omega$, $B' \subseteq B$ in \mathbb{P}_I , such that

$$\forall x \in B' : \neg \Phi(f(x), g(x)).$$

Since the last statement is Π_1^1 , it is preserved in generic extensions. In particular, $B' \Vdash \neg \Phi(f(x_{gen}), g(x_{gen}))$, whereas $B \Vdash \Psi(f(x_{gen})) = \forall y \Phi(f(x_{gen}), y)$ – a contradiction.

For Σ_2^1 provably ccc σ -ideals, we can add transcendence over L to the list of equivalent conditions. This is no more than adapting [14] Theorem 4.3 to our context, with a slight change in statement and almost no change in the proof.

Theorem 3.3. *Let I be a Σ_2^1 provably ccc σ -ideal. Then the following are equivalent:*

- (1) Δ_2^1 sets are I -measurable.
- (2) For every $B \in \mathbb{P}_I$ and for every real z , there is a \mathbb{P}_I -generic over $L[z]$ in B .

Proof. For $(1) \Rightarrow (2)$, we already know that (1) implies Σ_3^1 \mathbb{P}_I -generic absoluteness. The set of $L[z]$ generics in B is $\Pi_2^1(z)$ in this case, and can be forced to be nonempty. The conclusion follows.

As to $(2) \Rightarrow (1)$, using corollary 2.7 we may assume there is a real z such that $\omega_1^{L[z]} = \omega_1$. Let A be $\Delta_2^1(a)$ and $B \in \mathbb{P}_I$. For ease of notation, let us assume that z , a and B are all constructible, so that we can work in L and assume $\omega_1^L = \omega_1$.

We now decompose both $B \cap A$ and $B - A$ into \aleph_1 Borel sets, as both are Σ_2^1 sets. The decomposition is absolute between L and \mathbb{V} , since they both agree on the first uncountable ordinal. In particular, all those Borel sets are constructible. By assumption, there is a generic over L in B , which is, one of those Borel sets has an element which is L generic. It follows that this set is I -positive in L . Our definability assumption on I obligates it to be I -positive in \mathbb{V} as well, and the proof is completed. \square

We conclude the section with two remarks on \mathbb{P}_I -Borel uniformization.

Remark 3.4. The notion of \mathbb{P}_I -Borel uniformization is related to the notion of Borel canonization of Kanovei, Sabok and Zapletal [16]:

If Π_2^1 has \mathbb{P}_I -Borel uniformization then there is Borel canonization of analytic equivalence relations.

Proof. We use the rank defined in [7] section 3. Let $(x, f) \in A$ if and only if $f \in WO$ and $\delta(x) \leq f$. A is Π_2^1 . Since all classes are Borel, the sections of A are nonempty, so we can use \mathbb{P}_I -Borel uniformization and find $B \in \mathbb{P}_I$ and $f : B \rightarrow WO$ Borel such that $\delta(x) \leq f(x)$. The boundedness theorem completes the argument. \square

Remark 3.5. The following are equivalent:

- (1) Σ_3^1 - \mathbb{P}_I -generic absoluteness.
- (2) Given $\Phi(x, y, \bar{z})$ a Π_1^1 formula, the statement " $\Phi(x, y, \bar{z})$ has nonempty sections" is absolute between \mathbb{P}_I -generic extensions.
- (3) Given $\Phi(x, y, \bar{z})$ a Π_1^1 formula, the statement " $\Phi(x, y, \bar{z})$ is a graph of a function" is absolute between \mathbb{P}_I -generic extensions.

Proof. Above results and Π_1^1 uniformization. \square

4. FROM TRANSCENDENCE OVER L TO $PSP_I \Sigma_2^1$

In the following section we find transcendence properties over L which are sufficient conditions for $PSP_I(\Sigma_2^1)$.

Theorem 4.1. *Let I be Σ_2^1 or Π_2^1 and provably ccc. If for any real z and $B \in \mathbb{P}_I$ there is a perfect set $P \subseteq B$ of $\mathbb{P}_I * \dot{\mathbb{P}}_I$ generics over $L[z]$, then $PSP_I(\Sigma_2^1)$.*

We will say that I is *homogeneous* if \mathbb{P}_I is a weakly homogeneous forcing notion. The forcing notion [4] 2.6 has natural counterparts for any σ -ideal with the Fubini property, leading to the following corollary:

Corollary 4.2. *Let I be Σ_2^1 or Π_2^1 , provably ccc, homogeneous and with the Fubini property. If ω_1 is inaccessible to the reals then $PSP_I(\Sigma_2^1)$.*

See corollaries 5.4 and 5.5 for the application of theorem 4.1 on the meager and null ideals. Note that although the last corollary can be applied to those ideals, it does not produce any new result – when ω_1 is inaccessible to the reals, Σ_2^1 sets have the Baire property and are Lebesgue measurable, and Mycielski's theorems 1.2 and 1.3 are valid.

Proof. (of theorem 4.1) Let E be a Σ_2^1 equivalence relation on B Borel I -positive with I -small classes. We may assume E is lightface Σ_2^1 and B is constructible.

We first claim that the generic added by forcing with B belongs to a new E -class. Otherwise, fix $z \in \mathbb{V}$ and $B' \subseteq B$ such that

$$B' \Vdash x_G \in [z].$$

Let M be an elementary submodel of the universe containing z and all the relevant information. Let $x \in B'$ be M -generic. Then $M[x] \models xEz$, and so by Π_1^1 absoluteness, $\mathbb{V} \models xEz$. We have thus shown that all the M -generics in B' are in the equivalence class of z , hence $[z]$ is I -positive – a contradiction.

Consider the two-step iteration $\mathbb{P}_I * \dot{\mathbb{P}}_I$.

Claim 4.3. If $B_1 \subseteq B$ and $L \models B_1 \Vdash \dot{B}_2 \subseteq B$ then $L \models (B_1, \dot{B}_2) \not\Vdash (z_1 E z_2)$, where z_1, z_2 are the \mathbb{P}_I -generics.

Proof. The idea is similar to the one of the proof of [10] theorem 3.4. Note that the first generic we will mention is an L -generic that is an element of \mathbb{V} , while the second one is a real \mathbb{V} -generic.

Assume otherwise, and let $(B_1, \dot{B}_2) \in \mathbb{P}_I * \dot{\mathbb{P}}_I$ be as above and such that $L \models (B_1, \dot{B}_2) \Vdash z_1 E z_2$. Let

$$z_1 \in \mathbb{V}$$

be \mathbb{P}_I -generic over L such that $z_1 \in B_1$. Then $\dot{B}_2[z_1]$, the interpretation of \dot{B}_2 by the generic filter of z_1 , is an I -positive Borel set in $L[z_1]$ – we denote it by B_2 – which is a subset of B . Let $z_2 \in B_2$ be \mathbb{P}_I generic over \mathbb{V} . Then z_2 is also \mathbb{P}_I generic over $L[z_1]$. By the assumption

$$L[z_1][z_2] \models z_1 E z_2$$

and hence $\mathbb{V} \models z_1 E z_2$. However, we have shown that the \mathbb{P}_I -generic $z_2 \in B$ cannot be an element of the ground model set $[z_1]_E$ – a contradiction. \square

It follows that in L ,

$$D = \{(B_1, \dot{B}_2) : \neg(B_1 \subseteq B \wedge B_1 \Vdash \dot{B}_2 \subseteq B) \text{ or } (B_1, \dot{B}_2) \Vdash \neg(z_1 E z_2)\}$$

is dense in $\mathbb{P}_I * \dot{\mathbb{P}}_I$. Therefore, given $(x, y) \in B^2$ which is $\mathbb{P}_I * \dot{\mathbb{P}}_I$ generic over L ,

$$L[x][y] \models \neg(x E y)$$

which together with Shoenfield's absoluteness implies that x and y are inequivalent. Since we assumed there is a perfect set $P \subseteq B$ of $\mathbb{P}_I * \dot{\mathbb{P}}_I$ generics over L , that concludes the proof. \square

5. FROM $PSP_I \Sigma_2^1$ TO TRANSCENDENCE OVER L

In the following section we find necessary conditions for $PSP_I(\Sigma_2^1)$ and $PSP_I(\Delta_2^1)$, for I a Σ_2^1 and provably ccc σ -ideal. The author wishes to thank Amit Solomon for his help with obtaining the following two results.

Theorem 5.1. *Let I be Σ_2^1 and provably ccc. $PSP_I(\Sigma_2^1)$ implies that for every $B \in \mathbb{P}_I$ and for every real z there exists a perfect set $P \subseteq B$ of \mathbb{P}_I -generics over $L[z]$.*

Proof. Since $PSP_I(\Sigma_2^1)$ clearly implies $PSP_I(\Delta_2^1)$ and $PSP_{countable}(\Delta_2^1)$, theorem 1.7 and [13] guarantees that Sacks forcing preserves Σ_3^1 statements. A perfect set in B of \mathbb{P}_I -generics over L exists iff

$$\exists P \subseteq B \text{ perfect } (\forall x \in P \forall c ((c \in L) \wedge (c \in BC) \wedge (B_c \in I) \Rightarrow x \notin B_c).$$

That is a Σ_3^1 statement, hence if Sacks forcing adds a perfect subset of B of \mathbb{P}_I -generics, we are done.

We will find a condition $P \subseteq B$ in Sacks forcing such that any new real added to P must be \mathbb{P}_I -generic. Since Sacks forcing adds a perfect set of new reals to the condition P , that will be enough.

The first stage is defining a Σ_2^1 equivalence relation on B whose classes are either I -small sets in L , or singletons which are \mathbb{P}_I -generic elements over L .

Let $I_L(c)$ be the statement

$$(c \in L) \wedge (c \in BC) \wedge (B_c \in I).$$

Let $D(x, c)$ be the statement: $I_L(c)$, $x \in B_c$ and

$$\forall c' I_L(c') \wedge x \in B_{c'} \Rightarrow (c \leq_L c'),$$

which is, c is the 1st I -small set in L that has x as one of its elements. Note that $D(x, c)$ is Σ_2^1 since it can be decided inside a large enough countable model. We then consider the following Σ_2^1 equivalence relation on B :

$$\forall x, y \in B : x E y \Leftrightarrow (x = y) \vee \exists c (D(x, c) \wedge D(y, c)).$$

Under E , the \mathbb{P}_I -generics over L form equivalence classes that are singletons. The rest of the classes are all contained in an I -small set of L , hence are I -small. Since all classes are I -small, $PSP_I(\Sigma_2^1)$ implies the existence of a perfect set $P \subseteq B$ of pairwise inequivalent elements.

We first show that any new Sacks real in P must belong to a new E -class. By Shoenfield's absoluteness, P remains a perfect set of pairwise inequivalent elements in the \mathbb{P}_I -generic extension. In addition, if the class of $z \in \mathbb{V}$ had no representative in P -

$$\forall x \ x \in P \rightarrow \neg(xEz)$$

- then $P \cap [z]_E$ will remain empty in the generic extension as well. Therefore the new Sacks real in P must indeed belong to a new class.

We can now complete the proof by showing a Sacks real in P is \mathbb{P}_I -generic over L . Indeed, if it hadn't been, there would be $c \in L$ such that $D(x_{gen}, c)$, so after forcing

$$\exists y \in BD(y, c).$$

That is a Σ_2^1 statement, therefore true in the ground model as well. It means that x_{gen} is an element of a ground model class, which is a contradiction. \square

Theorem 5.2. *Let I be Σ_2^1 and provably ccc. $PSP_I(\Delta_2^1)$ implies that for every $B \in \mathbb{P}_I$ and for every real z there exists a \mathbb{P}_I -generic over $L[z]$ in B .*

Proof. Let D and E be defined as in the previous proof, and ZFC^* a large enough finite fragment of ZFC . We define another equivalence relation which is Π_2^1 : For $x, y \in B$, xFy if and only if

$$\forall M \ ((M \in WO) \wedge (M \models ZFC^*) \wedge (x, y \in M) \wedge (\exists c \ M \models D(x, c)) \Rightarrow (M \models D(y, c)))$$

and

$$\forall M \ ((M \in WO) \wedge (M \models ZFC^*) \wedge (x, y \in M) \wedge (\exists c \ M \models D(y, c)) \Rightarrow (M \models D(x, c))).$$

If x and y are not \mathbb{P}_I -generics over L , then $xEy \Leftrightarrow xFy$. The equivalence relations E and F are only different on the set of the \mathbb{P}_I -generics over L : under E , the \mathbb{P}_I -generics over L form equivalence classes that are singletons, whereas under F they form one equivalence class.

By way of contradiction, assume that in B there are no \mathbb{P}_I -generics over L . Then E and F coincide, and E becomes Δ_2^1 . $PSP_I(\Delta_2^1)$ then guarantees the existence of a perfect set P of pairwise inequivalent elements. We continue just as before – recall that for the Sacks Σ_3^1 generic absoluteness we only used $PSP_I(\Delta_2^1)$. We get a perfect set $P \subseteq B$ of \mathbb{P}_I -generics over L – a contradiction. \square

We can finally answer problem 1.5 (1) for a wide class of ccc σ -ideals:

Corollary 5.3. *Let I be Σ_2^1 and provably ccc. The following are equivalent:*

- (1) $PSP_I(\Delta_2^1)$.
- (2) Δ_2^1 sets are I -measurable.

Proof. (1) \Rightarrow (2) is the previous theorem together with proposition 3.3. (2) \Rightarrow (1) is proposition 3.2 and theorem 1.6. \square

However, problem 1.5 (2) has a negative answer:

Corollary 5.4. *The following are equivalent:*

- (1) $PSP_{meager}(\Sigma_2^1)$.

(2) Δ_2^1 sets have the Baire property.

Proof. Recall that for $I = meager$, if there is a \mathbb{P}_I generic over $L[z]$, then for every $B \in \mathbb{P}_I$ there is a perfect set $P \subseteq B$ of $\mathbb{P}_I * \dot{\mathbb{P}}_I$ generics over $L[z]$ (see [4] 1.1). \square

We summarize what we know about the case of the null ideal:

Corollary 5.5. *Let I be the null ideal:*

- (1) $PSP_I(\Sigma_2^1)$ implies the existence of a perfect set of random reals over $L[z]$, for any real z .
- (2) If for any real z there is a perfect set of $\mathbb{P}_I * \dot{\mathbb{P}}_I$ random reals over $L[z]$, then $PSP_I(\Sigma_2^1)$.

Proof. Random real forcing is weakly homogeneous, so random reals if exist, exist in every Borel set of positive measure. \square

If the existence of a perfect set of random reals is equivalent to the existence of a perfect set of mutually random reals then both conditions above are equivalent – that is an open question, see [4] 2.8.

We do not know the status of problem 1.5 (2) for the case of the null ideal - can we have $PSP_{null}(\Sigma_2^1)$ with a Σ_2^1 set which is not Lebesgue measurable?

Problem 5.6. Is it consistent to have a perfect set of $\mathbb{P}_I * \dot{\mathbb{P}}_I$ random reals over $L[z]$ for any real z , and a Σ_2^1 set which is not Lebesgue measurable?

Although the countable ideal is not ccc and hence out of the scope of the last 2 sections, we still find ourselves very curious about understanding $PSP_{countable}(\Sigma_2^1)$. In light of theorem 1.7 and [5] 7.1, we conjecture that:

Problem 5.7. If for every z there is $x \notin L[z]$, then $PSP_{countable}(\Sigma_2^1)$.

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